On Calabi–Yau manifolds: bridging enumerative geometry and string theory

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The goal of the talk was to call attention to one of the recent interactions between geometry and theoretical physics: ideas, results and problems pertaining to the enumerative geometry of rational curves (mainly on a Calabi–Yau manifold) were introduced and their significance discussed with regard to puzzling calculations in superstring theory.

Introduction

Let me begin by outlining what enumerative geometry deals with.[1] Ever since its formal beginning in the 1860's, the goal of this branch of algebraic geometry has been to solve problems that ask for the number of figures of a given kind that satisfy a given list of conditions (see Schubert [1879]).

The evolution of enumerative geometry is closely tied to the evolution of algebraic geometry: while algebraic geometry often has mustered inspiration in enumerative geometry for the introduction of concepts or the advancement of conjectures, it is also the case, in the opposite direction, that enumerative geometry has become enriched each time algebraic geometry has made progress in its conceptual foundation.

Schubert's enumerative calculus is a good example of the influence in the direct sense. Schubert introduced a 'symbolic calculus,[2] which allowed him to solve hosts of enumerative problems.[3] Schubert's symbolic calculus great success could not hide, however, that it lacked a solid mathematical foundation, and this was the main reason that led Hilbert to include the development of such a foundation as the problem number 15 in the list he compiled for the 1900 International Congress of Mathematicians. Problem 15, in its turn, was such a powerful stimulus for algebraic geometry that many concepts and fundamental results introduced after 1900 have their origin in it.

Conversely, the current foundation for enumerative geometry is the algebraic geometry of the day, especially intersection theory, but of course often it has to develop its own methods to meet its ends.

Among the figures studied by Schubert, rational curves occupy a distinguished place. In addition to lines and conics, he considered planar singular cubics (cuspidal cubics and nodal cubics), which are unexpectedly difficult to handle from an enumerative point of view, and also twisted cubics in $\mathbf{P}^3$, which by definition are curves C of $\mathbf{P}^3$ not contained in a plane and for which there exists a surjective map $\mathbf{P}^1 \to C$ given by cubic polynomials. It is not hard to see that any twisted cubic is projectively equivalent to the cubic given by the map

$$[t_0, t_1] \mapsto [t_0^3, t_0^2 t_1, t_0 t_1^2, t_1^3] .$$

For example, Schubert determined the characteristic numbers of the family of twisted cubics. These numbers involve the following three 'characteristic' conditions: ν, that the cubic meets a given line; ρ, that the cubic is tangent to a given plane; and P, that the cubic goes through a given point. These characteristic numbers were verified in Kleiman–Strømme–Xambó [1987]. On the other hand, Schubert results and methods for cuspidal and nodal plane cubics have been updated and extended in works such as Miret–Xambó [1989, 1991].

Let us now turn to the input coming from theoretical physics. In the last years there has been an enormous amount of interesting work related to string theories that is orders of magnitude beyond the control of experiment. It appears, nevertheless, that they are able to produce highly non-trivial and precise mathematical conjectures which so far seem very hard to prove rigorously (in the mathematical sense). In this situation it happens (see Jaffe–Quinn [1993]) that a (mathematical) proof of any of those statements plays the role that in traditional physics is played by experiments. Thus mathematical reasoning has become, in some sense, the 'mathemathron' of string theorists.[4]

Among the conjectures here we are going to consider only those that predict the number of rational curves on certain Calabi–Yau varieties, for example on a general quintic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$. String theorists calculate a 'Yukawa coupling' series $f(q)$ in two different ways, using a principle called 'mirror symmetry', and get the following two expressions:

$$f(q) = 5 + 2875q + 4876875q^2 + ...$$

and

$$\begin{aligned} f(q) &= 5 + \sum_{k \geq 1} n_k k^3 \frac{q^k}{1 - q^k} \\ &= 5 + n_1 q + (2^3 n_2 + n_1) q^2 + ... \end{aligned}$$

where n_k is the number of rational curves of degree k in the quintic threefold. The second expression comes, roughly speaking, from a quantum correction called 'sum over instantons' (which here we may take to mean rational curves). The values gotten for the first four n_k are the following:

k	n_k
1	2875
2	609 250
3	317 206 375
4	242 467 530 000

In Candelas et al. [1991], the work where such numbers were published for the first time, there is a table for $1 \leq k \leq 10$, and in principle string theorists can calculate n_k up to any value of k because it is possible to calculate, in principle, as many terms of the first form of the q-expansion of $f(q)$.

This result is very striking, even if we disregard the magic of such calculations and do not bother about what is the precise meaning of rational curves which is used. Indeed, if we write a_k for the k-th coefficient of the first form of the q-expansion, then $a_2 - a_1$ must be an integer divisible by 8, $a_3 - a_1$ must be and integer divisible by 27, and so on, which are unlikely properties at the very least. In fact there is no known a priori reason for the a_k to be integers and proving that they indeed are so would be very interesting.[5]

There is another reason why the results are so striking. They tacitly say that on a general quintic threefold there are only finitely many rational curves for each degree $k \geq 1$. This was in fact conjectured by Clemens (see Clemens [1983, 1984]), but its truth is far from known at present: Katz [1986 b] showed that the conjecture is correct for $k \leq 7$ and recently the cases $k = 8$ and $k = 9$ have been settled by Johnsen and Kleiman [1993]. (These authors show that there are finitely many irreducible reduced rational curves on a general quintic 3-fold of degree $k \in \{8, 9\}$, that its number is positive, and that they are necessarily smooth. For comparison let us add that Clemens' conjecture states not only that on a general quintic threefold there are finitely many rational curves, but also that each such curve is smooth, that *its normal bundle is* $\mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)$, where $\mathcal{O}_X(-1)$ is the tautological line bundle on $\mathbf{P}^1_{\mathbf{C}}$, and that *two distinct rational curves do not intersect*.)

But matters are, as they stand, far much worse when it comes to calculate the numbers n_k. The good news are that $n_1 = 2875$ was already known by Schubert over a century ago (it has been worked out using modern methods by many authors; see, for example, Katz [1983], or Xambó-Descamps [1993 b]); that n_2 had been calculated by S. Katz [1986 b, 1988], and confirmed by other authors (see next reference); and that the number n_3 has been calculated, after the string theorists discovery, by Ellingsrud–Strømme [1991].[6] The bad news are that for $k \geq 4$ nobody seems to know a sensible line of attack, much as it happened with the case $k = 3$ in the mid eighties. The principle behind the first computations (recently other methods have been discovered; we will come back to this later on) was basically the same, namely, to express the rational curves on the quintic threefold of a given degree $k \in \{1, 2, 3\}$ as the zero locus of a section of a suitable vector bundle on the variety parameterizing the curves in question (the rank of the bundle coincides with the dimension of the variety), but of course the complexity of the calculations increases dramatically from $k = 1$ to $k = 3$, since the varieties and bundles involved have dimensions and ranks 6, 11 and 16, respectively. Unfortunately the methods used so far look hopeless for $k \geq 4$ because the variety that parameterizes rational curves of degree 4 in $\mathbf{P}^4$ is conjectured to be singular and it is not clear how to go around this fact (see Piene [1993]; for other calculations of a similar nature, see Katz [1992, 1993] and Ellingsrud–Strømme [1993]).

Since the coefficients of $f(q)$ computed so far are indeed integers, and the resulting values of the n_k are also integers, and since for $k \in \{1, 2, 3\}$ the values coincide with those supplied by the mathemathron, it looks as if something very deep is going on with such calculations. What is it?

In section 1 we review the Hodge cohomology for a compact Kähler complex manifold; in section 2 we introduce, after Cecotti [1990, 1991] and Morrison [1993], the Yukawa couplings on a Calabi–Yau manifold; and in section 3 we attempt to sketch an introduction to the philosophy of string theory and also to the mirror symmetry principle.

1. Notations, terminology and generalities

1.1. For a compact Kähler manifold X,[7] we set T_X to denote the holomorphic tangent bundle, $\Omega_X = T_X^*$ to denote the locally free sheaf of holomorphic 1-forms on X and $\Omega_X^p = \Lambda^p \Omega_X$ the sheaf of holomorphic p-forms. If n is the dimension of X, then $\mathcal{K}_X = \Omega_X^n$ is called the *canonical sheaf*. By $H^{p,q}(X)$ we denote the cohomology space $H^q(X, \Omega_X^p)$. Its dimension, which is finite, will be denoted $h^{p,q}(X)$, or just $h^{p,q}$.

The first Chern class of Ω_X, which coincides with the first Chern class of $\mathcal{K}_{\mathcal{X}}$, is called the *canonical class* of X. From the definition of Ω_X it follows that the canonical class is $-c_1(X)$, where we set $c_i(X) = c_i(T_X)$ to denote the Chern classes of X.

1.2. There is a *Hodge decomposition*

$$H^k(X, \mathbf{C}) \simeq \bigoplus_{\substack{p+q=k \\ p,q \geq 0}} H^{p,q}(X)$$

This decomposition follows from the Hodge theorem for the de Rham cohomology,

$$H^k_{\mathrm{dR}} \simeq \bigoplus_{\substack{p+q=k \\ p,q \geq 0}} H^{p,q}_{\bar{\partial}}(X) \ ,$$

and the Dolbeault isomorphism

$$H^{p,q}_{\bar{\partial}}(X) \simeq H^q(X, \Omega_X^p) = H^{p,q}(X) \ .$$

Since $H^{q,p}(X) = \overline{H^{p,q}(X)}$, where the bar denotes complex conjugation, we see that $h^{q,p}(X) = h^{p,q}(X)$. On the other hand, there is a natural pairing

$$H^{i,j}(X) \times H^{n-i,n-j}(X) \to \mathbf{C} \ ,$$

given by integrating along X the exterior product, which leads to an isomorphism $H^{i,j}(X) \simeq H^{n-i,n-j}(X)^*$ (*Hodge duality*). Hence in particular

$$h^{i,j}(X) = h^{n-i,n-j}(X) \ .$$

Notice also that $h^{0,0}(X) = h^{n,n}(X) = 1$. The numbers $h^{i,j}$ are often arranged on the *Hodge diamond*. Since the numbers $h^{i,j}$ vanish if i or j is negative or greater than n, it looks like (say for 3-folds) the following:

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & \vdots & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & \cdots & h^{21} & \cdot\vdots\cdot & h^{12} & \cdots & h^{03} \\
 & h^{31} & & h^{22} & & h^{13} & \\
 & & h^{32} & \vdots & h^{23} & & \\
 & & & h^{33} & & &
\end{array}$$

By what we have just said, only the numbers in, say, the left top quarter (h^{10}, h^{20}, h^{30}, h^{11} and h^{21}) are independent, for the diamond is symmetrical with respect to the middle horizontal line (by the duality relation) and also with respect to the middle vertical line (by complex conjugation).

1.3. In particular we have that

$$b_k(X) = \sum_{i=0}^{k} h^{i,k-i}(X) ,$$

where $b_k(X)$, called the k-th Betti number of X, is the dimension of $H^k(X, \mathbf{C})$. The *Euler characteristic* of X, $\chi = \chi(X)$, which by definition is the alternating sum $\sum_{i=0}^{2n}(-1)^i b_i(X)$, can thus be computed in terms of the Hodge numbers. For example, for a 3-fold one gets

$$(a) \qquad \chi(X) = 2 - 4h^{0,1} + 4h^{0,2} - 2h^{0,3} + 2h^{1,1} - 2h^{1,2} .$$

By the Gauss–Bonnet theorem, the Euler characteristic of X is also equal to $\int c_n(X)$.

1.4. The *Hodge filtration* of $H^k(X)$ is the sequence of subspaces $F^p H^k(X)$ defined by the relation

$$F^p H^k(X) = \bigoplus_{j \geq p} H^{j,k-j}(X) .$$

It is equivalent to the Hodge decomposition, for

$$H^{p,k-p}(X) = F^p H^k(X) / F^{p+1} H^k(X) .$$

2. Calabi–Yau manifolds

2.1. A *Calabi–Yau manifold* is a compact Kähler manifold with trivial canonical bundle and such that $h^{i,0}(X) = 0$ for $1 \leq i \leq n-1$.

The triviality of the canonical bundle is equivalent to the vanishing of the first Chern class of X, because $c_1(X) = -c_1(\omega_X)$. Moreover, since the existence of an isomorphism $\Omega_X^n \simeq \mathcal{O}_X$ implies the existence of an isomorphism $T_X \simeq \Omega_X^{n-1}$, we have, taking sections, $h^0(T_X) = h^{n-1,0} = 0$. So a Calabi–Yau manifold has no global non-zero holomorphic vector fields. Moreover, we also have that

$$(a) \qquad H^1(T_X) \simeq H^1(\Omega_X^{n-1}) = H^{n-1,1}(X)$$

Another remark is that $h^{n,0} = 1$, for

$$h^{n,0} = \dim_{\mathbf{C}} H^0(X, \Omega_X^n) = \dim_{\mathbf{C}} H^0(X, \mathcal{O}_X) = h^{0,0}(X) = 1 .$$

Hence the value at the four vertices of the Hodge diamond of a Calabi–Yau manifold is 1 and the remaining entries around the edge are 0. In particular we see that for a Calabi–Yau 3-fold there are only 2 independent Hodge numbers, which henceforth we will choose to be $h^{1,1}(X)$ and $h^{2,1}(X)$. By 1.3 (*a*) we see that the Euler characteristic of a Calabi–Yau 3-fold is

$$\chi(X) = 2\left(h^{1,1}(X) - h^{1,2}(X)\right) .$$

2.2. There is an 'intersection map'

$$I_{1,1} : S^n H^{1,1}(X) \to \mathbf{C}$$

given by taking exterior multiplication

$$I_{1,1} : S^n H^{1,1}(X) \to H^0(\omega_X)$$

followed by integration over X, $\int_X : H^0(\omega_X) \to \mathbf{C}$. Here S^n denotes the n-th symmetric power (which we may take because the exterior product of (1,1)-forms is commutative).

2.3. Now the exterior product pairing

$$(a) \qquad H^{p,q}(X) \otimes H^{n-1,1}(X) \to H^{p+n-1,q+1}(X) ,$$

and the fact that $\Omega_X^n \simeq \mathcal{O}_X$, yield a pairing

$$H^{p,q}(X) \otimes H^{n-1,1}(X) \to H^{p-1,q+1}(X) ,$$

which together with 2.1 (*a*) yields a map

$$H^1(T_X) \to \bigoplus_{p,q} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) .$$

This map is called, after Carlson et al. [1984], the *differential of the period map*.

If in (*a*) we take j times the factor $H^{n-1,1}(X)$, instead of just one, we would come up, reasoning in the same way, with a map

$$S^j H^1(T_X) \to \bigoplus_{p,q} \mathrm{Hom}\left(H^{p,q}(X), H^{p-j,q+j}(X)\right) .$$

In particular there is a map

$$S^n H^1(T_X) \to \mathrm{Hom}\left(H^{n,0}(X), H^{0,n}(X)\right) .$$

But it is clear that

$$\mathrm{Hom}\left(H^{n,0}(X), H^{0,n}(X)\right) \simeq H^{n,0}(X)^* \otimes H^{0,n}(X) \simeq \left(H^{n,0}(X)^*\right)^{\otimes 2}$$

and so we finally get a map

$$(b) \qquad S^n H^1(T_X) \to (H^{n,0}(X)^*)^{\otimes 2}$$

which is called the *unnormalized Yukawa coupling* (or the *n-point Yukawa coupling*).

Since $H^{n,0}(X) = H^0(\Omega_X^n) \simeq H^0(\mathcal{O}_X) = \mathbf{C}$, $(H^{n,0}(X)^*)^{\otimes 2}$ is really a 1-dimensional vector space. Taking an isomorphism with $\mathbf{C}$ (a 'normalization'), we get a *normalized Yukawa coupling*. Such a normalization is tantamount to the choosing of a non-zero element of $H^{n,0}(X)^{\otimes 2}$, which is the dual space of the right hand side of (*b*).

2.4. Let $\mathcal{M}$ be the versal deformation space of a Calabi–Yau manifold X, and let $\xi = [X]$ be the point on $\mathcal{M}$ corresponding to X. Then it is known from general principles of deformation theory that the tangent space to $\mathcal{M}$ at $[X]$, $T_{\mathcal{M},\xi}$, is canonically isomorphic to $H^1(T_X)$. Since this is isomorphic to $H^1(\Omega_X^{n-1})$, and $\mathcal{M}$ is known to be smooth (see Morrison [1993] for references), we may conclude that there is an isomorphism

$$H^{n-1,1}(X) \simeq H^1(T_X) \simeq T_{\mathcal{M},\xi}$$

and that

$$\dim(\mathcal{M}) = h^{n-1,1}(X) \ .$$

In other words, $H^{n-1,1}(X)$ is the space of infinitesimal deformations of the complex structure of X. Notice that $H^{1,1}(X)$ can be regarded as the space of infinitesimal deformations of the Kähler form of X. These two spaces look completely unrelated from the geometrical point of view.

3. Strings and mirror symmetry

String theory is a vast and imposing subject (see, for example, Green–Schwartz–Witten [1987], Kaku [1988, 1991], Castellani–D'Auria–Fré [1991] and the large lists of references in them, especially in the first quoted book). Its goal is to provide a unified theory of the four known fundamental interactions. It starts postulating that the basic unit of energy (which we know, since Einstein theory of relativity, to be the same thing as matter) is a string-like, rather than point-like, and that the different particles are just quantum states (energy levels) of such strings. This view is interesting in that it gives from the start a good qualitative explanation of the known features of particles and interactions, especially if we admit that strings can break and recombine.

Since strings have to give rise to the smallest of the known particles (quarks), its dimensions must be very small. It turns out that they must be of the order of 10^{-33} cm (Planck's scale), which is about 15 orders of magnitude beyond the length scale that can be probed with present day techniques. In any case, the only known theoretical guide to deal with such entities is quantum mechanics. Thus theorists deal with the dynamics of a string by applying a variant of the Feynmann integral, one of the standard ways to present quantum mechanics. According to this approach, the probability for a transition from a state into another is expressed by an the integral of a 'density' over the space of all surfaces swept out by a possible evolution in space-time of the string, from the first state to the second, the density being of such a nature that the weight of a given surface is maximum for the classical (Lagrangian) evolution and falls off quickly to zero when the surface deviates more and more from the classical one.

Now it turns out, when realistic conditions are imposed (heterotic string, say), that a string theory can be consistent only in dimension 10. Theoreticians interpret this by saying that at low energy 6 of the ten dimensions are 'compactified'. One way to interpret this is that the 10-dimensional manifold is of the form $M \times X$, where M is Minkowski's space-time and X is a compact 6-dimensional manifold whose radius is of the order of Planck's scale (so it would be 'visible' only after increasing the resolution power of the present day gear about 15 orders of magnitude). Furthermore, in Candelas–Horowitz–Stromminger–Witten [1985] (see also Stromminger–Witten [1985]) it was shown that X must be a Calabi–Yau 3-fold, and so far any Calabi–Yau 3-fold is equally eligible. Thus the situation is that before compactifying the theory of the heterotic string is basically unique, but that its low energy behavior depends on a manifold X of which we only know that it is a Calabi–Yau 3-fold. It is possible to impose further conditions to this manifold, one of them being through the 'number of generations' of particles. We refer to Gepner [1987] for an introduction to this subject.

Now the physical content of X turns out to be an associated object called a 'superconformal quantum field theory', or $SCQFT$ for short (see Kaku [1988, 1991], Lüst–Theissen [1989], Cuerno–Sierra–Gómez [1991]). The relevant fields in this theory, which are grouped into 'fields' and 'antifields', are associated to $H^{1,1}$ and $H^{2,1}$. The fields and antifields, however, can be exchanged by 'supersymmetry', and so it is possible to interchange the physical role of $H^{1,1}$ and $H^{2,1}$. This observation led to the introduction of the 'mirror symmetry' concept (the term 'mirror symmetry' appears for the first time in Greene–Plesser [1990]; see also Green–Plesser [1991]). According to this principle, there should exist, given a Calabi–Yau 3-fold X, another Calabi–Yau 3-fold X' yielding the same physical $SCQFT$, but with reversed (hence the 'mirror' term) cohomologies.

Thus we define a Calabi–Yau 3-fold X' to be a *mirror of* of a Calabi–Yau 3-fold X if the Hodge diamond of X' is obtained by interchanging $h^{1,1}$ and $h^{2,1}$, that is, if

$$h^{1,1}(X') = h^{2,1}(X) \ , \quad h^{2,1}(X') = h^{1,1}(X) \ .$$

The 'mirror symmetry conjecture' asserts that given a Calabi–Yau 3-fold X, there exists a mirror X' of X such that the $SCQFT$'s associated to X and X' are the same. Thus the existence of mirrors was predicted on theoretical physics grounds. Soon after that, string theorists (and mathematicians) have computed thousands of mirror pairs and there is quick progress in this area.

Now we want to describe the non trivial relations gotten by the equivalent physical theories of a mirror pair. Let I be the cubic (intersection) form on $H^{1,1}$ and Y a normalized Yukawa coupling on $H^{2,1}$. Consider the 'quantum intersection form' I^Q on $H^{1,1}$ as follows:

$$I^Q(\alpha_1,\alpha_2,\alpha_3) = \sum_{[\mathbf{P}^1 \xrightarrow{\varphi} X]} n(\varphi)^{-3} e^{-\int \varphi^*\omega_X} \int \varphi^*\alpha_1 \int \varphi^*\alpha_2 \int \varphi^*\alpha_3 \ ,$$

where ω_X is a Kähler form on X and where $n(\varphi)$ is the covering degree of φ. With this terminology, the 'mirror symmetry principle' states that

$$I^Q(X) = Y(X') \ , \quad Y(X) = I^Q(X') \ .$$

Since the evaluation of $I^Q(X)$ can be expressed in terms of the rational curves on X (for the quintic 3-fold it turns out to be the second form of $f(q)$ in the introduction), the game becomes to calculate a mirror X' of X and its Yukawa coupling, and all this is usually done by using methods of variation of Hodge structure. However it goes, such computations have been carried out in many cases now, they all lead to predictions about the number of rational curves on a given X (and even other types of curves), and the numbers have been confirmed (as hinted in the introduction) in relatively small (but growing) number of cases.

For more details on the ideas discussed in this paper, see:

Aspinwall–Lütken–Ross [1990], Aspinwall–Morrison [1993], Hübsch [1992], Manin [1993], Xambó-Descamps [1994].

Notes

1. For a historical perspective centered on the 'principle of conservation of number, see Xambó-Descamps [1993 a].
2. Sometimes it is called *Schubert calculus*, but today this locution is mostly reserved for the application of *symbolic calculus* to enumerative problems concerning linear varieties.
3. For example, finding that there are: 2 lines in 3-space meeting 4 given lines; or 92 conics in 3-space meeting 8 given lines; or 128 plane cuspidal cubics whose cusp and flex lie each in a given line and in addition go through 3 given points and are tangent to 2 given lines; or 640 twisted cubics that go through 3 points, meet 4 given lines and are tangent to 2 given planes.
4. Since some speak of string theories as 'Theories Of Everything', should we say 'TOEcists'?
5. Morrison [1993], p. 238, writes: "These arguments have a rather numerological flavor. I am reminded of the numerological observations made by Thompson and [by] Conway and Norton about the j-function and the monster group. At the time [it was 1979] no connection between these two mathematical objects was known. The q-expansion [Fourier series] of the j-function was known to have integer coefficients, and it was observed that these integers were integral linear combinations of the degrees of the irreducible representations of the monster group. This prompted much speculation about possible deep connections between the two, but at the outset all such speculation had to be characterized as 'moonshine' (Conway and Norton's term). The formal similarities to the present work should be clear: a q-expansion of some kind is found to have integer coefficients, and these integers then appear to be linear combinations of another set of integers, which occur elsewhere in mathematics in a rather unexpected location. Perhaps it is too much to hope that the eventual explanation will be as pretty in this case."
6. For a quick and clear introduction to the geometry behind these and other calculations, see Piene [1993].
7. We make the convention that a Kähler manifold is equipped, as part of its definition, with a Kähler form $\omega = \omega_X$.

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Gromov-Witten theory is used to define an enumerative geometry of curves in Calabi-Yau 4-folds. The main technique is to find exact solutions to moving multiple cover integrals. The resulting invariants are analogous to the BPS counts of Gopakumar and Vafa for Calabi-Yau 3-folds. We conjecture the 4-fold invariants to be integers and expect a sheaf theoretic explanation. Several local Calabi-Yau 4-folds are solved exactly. This is the second of the series of articles on the geometry of String Theory compactifications. Before reading this note, the interested reader may want to read the first note, where the concept of compactification background is introduced in the context of String Theory and M-Theory compactifications. As it is well known, to be well-defined , String Theory and M-Theory require respectively a ten and a eleven dimensional space-time.Â referred as the external manifold. Given this set-up, the first thing we need to do in order to compactify the theory is to find which are the internal manifolds allowed by String Theory or M-Theory. Once we know the allowed manifolds, we can pick one of them and then perform an explicit compactification on that particular compactification background.